LESSON

# 6.3 Solving Systems with Inverse Matrices

**OBJECTIVES**

- Find the inverse of a matrix, if it exists
- Use inverse matrices to solve systems of equations
- Understand the characteristics of an identity matrix
- Rewrite systems of equations as matrix equations

Consider the following scenario: Three friends, Duane, Marsha, and Parker, decide to take their younger siblings to the movies. Before the movie, they buy some snacks at the concession stand. Duane buys two candy bars, a small drink, and two boxes of chocolate-covered peanuts for a total of \$11.85. Marsha spends \$9.00 on a candy bar, two small drinks, and one box of chocolate-covered peanuts. Parker spends \$12.35 on two small drinks and three boxes of chocolate-covered peanuts, but doesn't buy any candy bars. If all the prices include tax, what is the price of each item?

Set up a system of equations using the prices of the items as the unknowns. Let $c$ represent the price of a candy bar in dollars, let $d$ represent the price of a small drink in dollars, and let $p$ represent the price of a box of chocolate-covered peanuts in dollars. This system represents the three friends' purchases:

$$2c + d + 2p = 11.85$$
$$c + 2d + p = 9$$
$$2d + 3p = 12.35$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \\ p \end{bmatrix} = \begin{bmatrix} 11.85 \\ 9 \\ 12.35 \end{bmatrix}$$

*Solve each of the following with the multiplicative inverse.*

$\frac{1}{3} \cdot 3x = 12 \cdot \frac{1}{3}$

$x = 4$

$-\frac{7}{5} \cdot \frac{-5}{7}x = 35 \cdot -\frac{7}{5}$

$x = -49$

**Discuss with your partner or group how you know when a number is a multiplicative inverse and how you know what the multiplicative identity is.**

Inverse is the reciprocal.

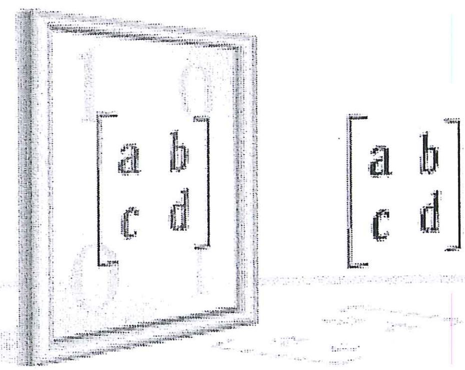
Multiplicative identity is 1 since any number multiplied by 1 remains unchanged.

### Identity Matrix

An **identity matrix**, symbolized by $[I]$, is the square matrix that does not alter the entries of a square matrix $[A]$ under multiplication.

$$[A][I] = [A] \text{ and } [I][A] = [A]$$

Matrix $[I]$ must have the same dimensions as matrix $[A]$, and it has entries of 1's along the main diagonal (from top left to bottom right) and 0's in all other entries.




Now that you know the identity matrix for a 2 × 2 matrix, you can look for a way to find the inverse of a 2 × 2 matrix.

### Inverse Matrix

The **inverse matrix** of $[A]$, symbolized by $[A]^{-1}$, is the matrix that will produce an identity matrix when multiplied by $[A]$.

$$[A][A]^{-1} = [I] \text{ and } [A]^{-1}[A] = [I]$$

$$\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} = \begin{bmatrix} 3+0 & 0+5 \\ 4+0 & 0+6 \end{bmatrix}$$

# Investigation • The Inverse Matrix

**Step 1** Use the definition of an inverse matrix to set up a matrix equation. Use these matrices and the $2 \times 2$ identity matrix for $[I]$.

$$[A] = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \qquad [A]^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \cdot A^{-1} = I$$

**Step 2** Use matrix multiplication to find the product $[A][A]^{-1}$. Set that product equal to matrix $[I]$.

$$\begin{bmatrix} 2a + c & 2b + d \\ 4a + 3c & 4b + 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Step 3** Use the matrix equation from Step 2 to write equations that you can solve to find values for $a$, $b$, $c$, and $d$. Solve the systems to find the values in the inverse matrix.

$2a + c = 1$

$4a + 3c = 0$

$c = 1 - 2a$

$4a + 3(1 - 2a) = 0$

$4a + 3 - 6a = 0$

$-2a + 3 = 0$

$\frac{3}{2} = \frac{2a}{2}$

$\boxed{1.5 = a}$

$c = 1 - 2(1.5)$

$= 1 - 3$

$\boxed{c = -2}$

$2b + d = 0$

$4b + 3d = 1$

$d = -2b$

$4b + 3(-2b) = 1$

$4b - 6b = 1$

$-2b = 1$

$\boxed{b = -\frac{1}{2}}$

$d = -2(-\frac{1}{2})$

$\boxed{d = 1}$

$$\boxed{A^{-1} = \begin{bmatrix} 1.5 & -.5 \\ -2 & 1 \end{bmatrix}}$$

**Step 4** Use your calculator to find $[A]^{-1}$. If this answer does not match your answer to Step 3, check your work for mistakes. [ See Calculator Note 6D to learn how to find the inverse on your calculator. ]

**Step 5** Find the products of $[A][A]^{-1}$ and $[A]^{-1}[A]$. Do they both give you $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$? Is matrix multiplication always commutative? No

$A \cdot A^{-1}$

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1.5 & -.5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3+-2 & -1+1 \\ 6+-6 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$I$

$A^{-1} \cdot A$

$$\begin{bmatrix} 1.5 & -.5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3+-2 & 1.5-1.5 \\ -4+4 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$I$

**Step 6** Not every square matrix has an inverse. Try to find the inverse of each of these matrices. Make a conjecture about what types of $2 \times 2$ square matrices do not have inverses.

a. $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 50 & -75 \\ 10 & -15 \end{bmatrix}$ c. $\begin{bmatrix} 10.5 & 1 \\ 31.5 & 3 \end{bmatrix}$ d. $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

None have inverses - A 2x2 matrix will not have an inverse when one row is a multiple of another.